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KKM – A TOPOLOGICAL APPROACH FOR TREES

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The Knaster–Kuratowski–Mazurkiewicz (KKM) theorem is a powerful tool in many areas of mathematics. In this paper we introduce a version of the KKM theorem for trees and use it to prove several combinatorial theorems.

A *2-tree hypergraph* is a family of nonempty subsets of $T \cup R$ (where T and R are trees), each of which has a connected intersection with T and with R. A *homogeneous 2-tree hypergraph* is a family of subsets of T each of which is the union of two connected sets.

For each such hypergraph $\mathcal{H}$ we denote by $\tau(\mathcal{H})$ the minimal cardinality of a set intersecting all sets in the hypergraph and by $\nu(\mathcal{H})$ the maximal number of disjoint sets in it.

In this paper we prove that in a 2-tree hypergraph $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$ and in a homogeneous 2-tree hypergraph $\tau(\mathcal{H}) \leq 3\nu(\mathcal{H})$. This improves the result of Alon [3], that $\tau(\mathcal{H}) \leq 8\nu(\mathcal{H})$ in both cases.

Similar results are proved for *d-tree hypergraphs* and *homogeneous d-tree hypergraphs*, which are defined in a similar way. All the results improve the results of Alon [3] and generalize the results of Kaiser [1] for intervals.

1. Introduction

Let $n \geq 0$ be an integer number. An n-dimensional *simplex* is the convex hull of $n+1$ points in general position in $\mathbb{R}^n$, called the *vertices* of the simplex. For $k = 0, \dots, n$, we define the *k-skeleton* of the simplex to be the union of all convex hulls of $k+1$ vertices. In this paper Δ_n will be the n-dimensional simplex, generated by $(0,0,\dots,0), (1,0,\dots,0), \dots, (1,1,\dots,1)$.

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This set of vertices will be denoted by G_n. Note that $(x_1, x_2, \dots, x_n)$ is in Δ_n if and only if $0 \le x_1 \le x_2 \le \dots \le x_n \le 1$. We write $I = \Delta_1 = [0,1]$.

The Knaster–Kuratowski–Mazurkiewicz (KKM) theorem (see [5]) says the following:

Theorem 1. *For every vertex g of Δ_n let A_g be an open set in Δ_n. If for each set W of vertices, the convex hull of W is contained in $\bigcup_{g \in W} A_g$ then*

$$\bigcap_{g \in G_n} A_g \neq \emptyset\,.$$

The theorem is also true if we use closed rather than open sets. One generalization of the KKM theorem is the following:

Theorem 2. *Let $d, n_1, \dots, n_d$ be positive integer numbers and let $S_1, \dots, S_d$ be d disjoint simplexes of dimensions $n_1, \dots, n_d$ respectively, and with sets of vertices $V_1, \dots, V_d$ respectively. For every $i = 1, \dots, d$ and every $g \in V_i$ let A_g be an open set in $\prod_j S_j$. Suppose that for each d sets $W_1 \subseteq V_1, \dots, W_d \subseteq V_d$, we have $\prod_i \mathrm{co}(W_i) \subseteq (\bigcup_i \bigcup_{g \in W_i} A_g)$. Then for some $i \in \{1, \dots, d\}$ we have*

$$\bigcap_{g \in V_i} A_g \neq \emptyset\,.$$

Here co is the convex hull of a set. Again, we can use closed rather than open sets.

The KKM theorem is a powerful tool in many areas of mathematics. Here is a useful corollary from the KKM theorem:

Corollary 1. *Let $f : \Delta_n \to \mathrm{L}^1(I)$ be a continuous function. Then there exist some $x = (x_1, x_2, \dots, x_n) \in \Delta_n$ with*

$$\left| \int_0^{x_1} f(x)(z)dz \right| = \left| \int_{x_1}^{x_2} f(x)(z)dz \right| = \dots = \left| \int_{x_n}^{1} f(x)(z)dz \right|.$$

An example of a simple theorem that can be proved either using this corollary or directly from the KKM theorem is Gallai's Theorem (see [4]). If X is a set and $\mathcal{H}$ is a family of nonempty subsets of X, then we say that $\mathcal{H}$ is a *hypergraph* on X. We write $\nu(\mathcal{H})$ for the maximal number of disjoint sets in $\mathcal{H}$, and $\tau(\mathcal{H})$ for the minimal cardinality of a subset of X intersecting all sets in $\mathcal{H}$. Note that we always have $\nu(\mathcal{H}) \le \tau(\mathcal{H})$. Gallai's Theorem says

Theorem 3. *Let $\mathcal{H}$ be a finite family of intervals in I, then $\tau(\mathcal{H}) = \nu(\mathcal{H})$.*

This theorem stays true if we replace I by a tree. In this paper, the word *tree* will be a short way to say "a topological space induced by a finite tree graph", that is, a compact, contractible, Hausdorff, topological space, where each point has a neighborhood homeomorphic to $\{z \in \mathbb{C} : z^n = |z^n|\}$ for some integer number $n \geq 1$.

Theorem 4. *Let T be a tree and let $\mathcal{H}$ be a finite family of connected subsets of T, then $\tau(\mathcal{H}) = \nu(\mathcal{H})$.*

Theorem 4 seems to be closely related to Theorem 3. However, all the proofs known so far for Theorem 4 are combinatorial. In this paper we will develop a method that will allow us to prove Theorem 4 using a generalization of the KKM theorem.

We will use this method to translate other theorems from the language of intervals to the language of trees. First we will prove a tree analog of Corollary 1, then we will use it to develop a tree analog of the methods established by Kaiser in [1].

Let d be a positive integer. A *d-interval hypergraph* is a family of nonempty subsets of $[0,1] \cup [2,3] \cup \cdots \cup [2d-2, 2d-1]$, each of which has a connected intersection with $[2i-2, 2i-1]$ $(i = 1, \ldots, d)$. A *homogeneous d-interval hypergraph* is a family of subsets of $[0,1]$, each of which is the union of d connected sets. Similarly, a *d-tree hypergraph* is a family of nonempty subsets of $T_1 \cup \cdots \cup T_d$ (where $T_1, \ldots, T_d$ are disjoint trees), each of which has a connected intersection with T_i $(i = 1, \ldots, d)$. A *homogeneous d-tree hypergraph* is a family of subsets of a tree T, each of which is the union of d connected sets.

In 1995 Tardos [2] proved

Theorem 5. *If $\mathcal{H}$ is a 2-interval hypergraph then $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$.*

This proof was simplified by Kaiser [1] and Aharoni (unpublished). Aharoni's proof uses Theorem 2. In [1], Kaiser also generalized the result:

Theorem 6. *If $\mathcal{H}$ is a d-interval hypergraph then $\tau(\mathcal{H}) \leq (d^2 - d)\nu(\mathcal{H})$. If $\mathcal{H}$ is a homogeneous d-interval hypergraph then $\tau(\mathcal{H}) \leq (d^2 - d + 1)\nu(\mathcal{H})$. This result can be improved to $\tau(\mathcal{H}) \leq (d^2 - d)\nu(\mathcal{H})$ if $d > 2$ and there is no projective plane of order $d - 1$.*

To be historically accurate, Kaiser used the Borsuk–Ulam theorem. However, Kaiser's proof of Theorem 6 does not use the entire strength of the Borsuk–Ulam theorem. The proof can easily be simplified with the same methods of Aharoni to a proof using the KKM theorem. This allows us to use the methods developed in this paper to prove

Theorem 7. *If $\mathcal{H}$ is a d-tree hypergraph then $\tau(\mathcal{H}) \leq (d^2 - d)\nu(\mathcal{H})$.*

Theorem 8. *If $\mathcal{H}$ is a homogeneous d-tree hypergraph then $\tau(\mathcal{H}) \leq (d^2 - d+1)\nu(\mathcal{H})$. This result can be improved to $\tau(\mathcal{H}) \leq (d^2 - d)\nu(\mathcal{H})$ if $d > 2$ and there is no projective plane of order $d-1$.*

It should be mentioned here that although we treat here trees as topological spaces, the results here are still valid for tree graphs. Each of the Theorems 3, 4, 5, 6, 7 and 8 has a discrete analogue, obtained merely by replacing topological trees by tree graphs, intervals by path graphs, and topological connectivity by graph connectivity. In each case, the discrete analogue is equivalent to the original theorem, in the sense that each of them can be proved easily from the other.

Note that the best known result so far for homogeneous d-tree hypergraph is $\tau(\mathcal{H}) \leq 2d^2\nu(\mathcal{H})$ by Alon [3]. This is also the best known result for d-tree hypergraph (for $d \geq 2$).

2. Notations

Let T be a tree. If $x \in T$ we write $T - x$ for $T \setminus \{x\}$. We say that x is a *leaf*, a *regular point* or a *singular point* if $T - x$ has one, two or more connected components, respectively. We write $\mathrm{L}(T), \mathrm{Reg}(T), \mathrm{Sin}(T)$ for the set of all leaves, regular points or singular points in T, respectively. Note that $|\mathrm{Sin}(T)| < |\mathrm{L}(T)| < \infty$ and $\mathrm{Reg}(T)$ is dense.

If $x, y \in T$ are two distinct points, let $\mathit{\Gamma}_{xy}$ be an embedding of I in T with $\mathit{\Gamma}_{xy}(0) = x$ and $\mathit{\Gamma}_{xy}(1) = y$.

Let $\lambda \geq 1$ be an integer number. We consider the symmetric power $\mathrm{Sym}^{\lambda}(T) = T^{\lambda}/S_{\lambda}$. That is, $\mathrm{Sym}^{\lambda}(T)$ is the topological space of all λ-tuples $(x_1, \ldots, x_{\lambda})$ (with product topology) where we identify $(x_1, \ldots, x_{\lambda})$ with $(x_{\pi(1)}, \ldots, x_{\pi(\lambda)})$ for every permutation π in the symmetric group S_{λ} (see Figure 1).

If $x = (x_1, \ldots, x_{\lambda}) \in \mathrm{Sym}^{\lambda}(T)$ we write $T - x$ for $T \setminus \{x_1, \ldots, x_{\lambda}\}$. We say that x is *a corner* if $x_1, \ldots, x_{\lambda}$ are all leaves. We write $\mathrm{Cor}(\mathrm{Sym}^{\lambda}(T))$ for the set of all corners of $\mathrm{Sym}^{\lambda}(T)$. Note that $\mathrm{Sym}^{\lambda}(I) = \mathit{\Delta}_{\lambda}$ and $\mathrm{Cor}(\mathrm{Sym}^{\lambda}(I)) = G_{\lambda}$.

If t is a subset of T then we define $\mathrm{Sym}^{\lambda}(t) = \{(x_1, \ldots, x_{\lambda}) \in \mathrm{Sym}^{\lambda}(T) : x_1, \ldots, x_{\lambda} \in t\}$. Note that $\mathrm{Sym}^{\lambda}(t)$ is an open/closed subset of $\mathrm{Sym}^{\lambda}(T)$ if and only if t is an open/closed subset of T.

If d is a positive integer number and $T_1, T_2, \ldots, T_d$ are trees, it is sometimes convenient to consider the disjoint union $F = T_1 \cup \cdots \cup T_d$ (having $T_1, T_2, \ldots, T_d$ as its connected component). If $\lambda_1, \ldots, \lambda_d$ are positive integer numbers, we may be interested in the Cartesian product $\prod_i \mathrm{Sym}^{\lambda_i}(T_i)$. For

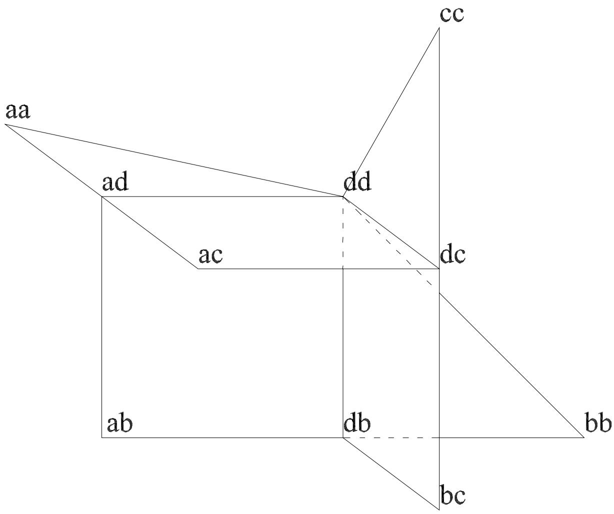

Figure 1. This is an illustration of $\mathrm{Sym}^2(T)$, where T is a tree shaped as the letter "T", that is, a tree with three leaves and one singular point.

a point $x=(x_1,\dots,x_d)=((x_{1,1},\dots,x_{1,\lambda_1}),\dots,(x_{d,1},\dots,x_{d,\lambda_d}))\in\prod_i \mathrm{Sym}^{\lambda_i}(T_i)$ we write $F-x=F\setminus\{x_{i,j}:i=1,\dots,d,\ j=1,\dots,\lambda_i\}$.

If $z\in T$ then we can define an embedding $\phi_z:\mathrm{Sym}^\lambda(T)\to\mathrm{Sym}^{\lambda+1}(T)$ by $\phi_z((x_1,\dots,x_\lambda))=(x_1,\dots,x_\lambda,z)$. For a mapping $f:T\to T$ we define $\tilde{f}:\mathrm{Sym}^\lambda(T)\to\mathrm{Sym}^\lambda(T)$ by $\tilde{f}((x_1,\dots,x_\lambda))=(f(x_1),\dots,f(x_\lambda))$. If f is continuous, so is $\tilde{f}$. Similarly, for a mapping $f:T\times I\to T$ we define $\tilde{f}:\mathrm{Sym}^\lambda(T)\times I\to\mathrm{Sym}^\lambda(T)$ by $\tilde{f}((x_1,\dots,x_\lambda),b)=(f(x_1,b),\dots,f(x_\lambda,b))$. Again, if f is continuous, so is $\tilde{f}$. Particularly, this means that since T is contractible, so is $\mathrm{Sym}^\lambda(T)$.

If $c,x\in\mathrm{Sym}^\lambda(T)$ and t is a subtree of T, we say that t *does not distinguish* c from x if c and x have the same number of elements in t (where, of course, multiplicities are taken into account). For $z\in T-x$, we say that z does not distinguish c from x if no connected component of $T-z$ does. (This happens if and only if c and x are in the same connected component of $\mathrm{Sym}^\lambda(T-z)$.) We define $E(c,x)$ to be the set of all points in $T-x$ that do not distinguish c form x. Note that although this definition is symmetric, we will usually use it when c is a corner. Let $C(c,x)$ be the union of all connected components of $T-x$ contained in $E(c,x)$.

Let now Y be any subset of $\mathrm{Sym}^\lambda(T)$. We define the *convex hull* of Y to be

$$\mathrm{co}_{T\lambda}(Y)=\left\{x:\bigcup_{y\in Y}C(y,x)=T-x\right\}.$$

In other words, $\mathrm{co}_{T\lambda}(Y)$ is the set of all $x \in \mathrm{Sym}^\lambda(T)$ with the property that for any component t of $T - x$ there exists some $y \in Y$ so that no point of t distinguishes y from x.

If T and/or λ are known then we may write $\mathrm{co}_T(Y)$, $\mathrm{co}_\lambda(Y)$, or $\mathrm{co}(Y)$ instead of $\mathrm{co}_{T\lambda}(Y)$. Note that if $\lambda = 1$ and $x, y \in T = \mathrm{Sym}^\lambda(T)$ are two distinct points then $\mathrm{co}(\{x,y\}) = \mathit{\Gamma}_{xy}(I)$, which is the path connecting x with y. In general for any finite $Y \subseteq T$, the set $\mathrm{co}_1(Y)$ is the minimal connected set containing Y. This is the usual definition of convex hull on a tree. Note also that if $Y \subseteq \mathrm{Cor}(\mathrm{Sym}^\lambda(I))$ then $\mathrm{co}_I(Y)$ is the convex hull of Y by the usual definition. However, if $Y \not\subseteq \mathrm{Cor}(\mathrm{Sym}^\lambda(I))$, this is not necessarily true.

3. Some easy lemmas

In this section we fix some tree T and some integer number $\lambda \geq 1$.

Lemma 1. $\mathrm{co}(\mathrm{Cor}(\mathrm{Sym}^\lambda(T))) = \mathrm{Sym}^\lambda(T)$.

Proof. Let $x = (x_1, \ldots, x_\lambda) \in \mathrm{Sym}^\lambda(T)$, and let t be a connected component of $T - x$. For each $i = 1, \ldots, \lambda$ let l_i be a leaf in a connected component of $T - x_i$ which does not include t (if x_i is a leaf, choose $l_i = x_i$). Then clearly no $z \in t$ distinguishes x from the corner $c = (l_1, \ldots, l_\lambda)$. Therefore $t \subseteq C(c, x)$. We can find such corner c for every connected component t of $T - x$, so $x \in \mathrm{co}(\mathrm{Cor}(\mathrm{Sym}^\lambda(T)))$. ∎

Lemma 2. *If $w, x, y \in \mathrm{Sym}^\lambda(T)$ then $E(w,x) \cap E(x,y) \subseteq E(w,y)$.*

Proof. This is the same as saying that for every $z \in T$ the relation of "not being distinguished by z" is transitive, which is the same as saying that the relation of "being in the same connected component of $\mathrm{Sym}^\lambda(T - z)$" is transitive. Clearly, it is. ∎

Corollary 2. *If $w, x, y \in \mathrm{Sym}^\lambda(T)$ then $(E(w,x) \setminus E(x,y)) \cup (E(x,y) \setminus E(w,x)) \subseteq T \setminus E(w,y)$.*

Lemma 3. *Suppose T has some bounded measure μ defined on it, so that open sets are measurable and finite sets have measure 0. Let $c \in \mathrm{Sym}^\lambda(T)$ and let $f : \mathrm{Sym}^\lambda(T) \to \mathrm{L}^1(T, \mu)$ be the function sending every $x \in \mathrm{Sym}^\lambda(T)$ to the characteristic function of $E(c, x)$. Then f is continuous.*

Proof. Let $x = (x_1, \ldots, x_\lambda) \in \mathrm{Sym}^\lambda(T)$ and $\varepsilon > 0$ be arbitrarily chosen. For every $i = 1, \ldots, \lambda$, since T is Hausdorff and compact and since $\mu(\{x_i\}) = 0$, we can find some neighborhood N_i of x_i with $\mu(N_i) < \varepsilon$. We may also demand that N_i contains no singular point other than x_i. Let $N_x = \{(y_1,\dots,y_\lambda) : (\forall i)\, y_i \in N_i\}$. Clearly, for every $y \in N_x$ we have $T \setminus E(x,y) \subseteq \bigcup_{i=1}^{\lambda} N_i$. Now by Corollary 2 we have $\|f(y)-f(x)\| \le \mu(T \setminus E(x,y)) < \lambda\varepsilon$. ∎

Note that Lemma 3 is not true if we use the characteristic function of $C(c,x)$ instead of $E(c,x)$.

Lemma 4. *Let T be a tree, λ a positive integer number, $c \in \mathrm{Sym}^\lambda(T)$ and $t \subseteq T$. Then*

i. *If t is open then $\{x \in \mathrm{Sym}^\lambda(T) : t \subseteq E(c,x)\}$ is closed.*
ii. *If t is closed then $\{x \in \mathrm{Sym}^\lambda(T) : t \subseteq E(c,x)\}$ is open.*
iii. *If t is open and connected then $\{x \in \mathrm{Sym}^\lambda(T) : t \subseteq C(c,x)\}$ is closed.*

Proof. The set $\{x \in \mathrm{Sym}^\lambda(T) : t \subseteq E(c,x)\}$ is the connected component of $\mathrm{Sym}^\lambda(T \setminus t)$ containing c (or $\emptyset$ if c is not there). This gives i and ii.

Assume now the conditions of iii and let $X = \{x \in \mathrm{Sym}^\lambda(T) : t \subseteq C(c,x)\}$. We need to show that every $x \in \mathrm{Sym}^\lambda(T) \setminus X$ has a neighborhood outside of X. Fix such x. If $t \not\subseteq E(c,x)$, take $\{y \in \mathrm{Sym}^\lambda(T) : t \not\subseteq E(c,y)\}$, which is open by i. Otherwise, $t \subseteq E(c,x)$ but $t \not\subseteq C(c,x)$, so there is a point z_1 in the same connected component of $T - x$ as t, that distinguishes x from c. Let Z be the closed path connecting z_1 with some point $z_2 \in t$. Consider the connected component N_x of $\mathrm{Sym}^\lambda(T \setminus Z)$ containing x. For every $y \in N_x$, the point z_1 distinguishes y from c, so the connected component of $T - y$ containing Z is not contained in $E(c,y)$. This implies that $z_2 \notin C(c,y)$, so N_x is really a neighborhood of x outside X, so X is closed. This gives iii. ∎

Note that part iii of the Lemma isn't true is we don't demand that t is connected. Also, if t is closed and connected, $\{x \in \mathrm{Sym}^\lambda(T) : t \subseteq C(c,x)\}$ isn't necessarily open. The last two lemmas demonstrate the abrupt behavior of $C(c,x)$ in comparison with $E(c,x)$. In general, proofs are more likely to work if they use $E(c,x)$ rather than $C(c,x)$. The following lemma sometimes helps in doing that.

Lemma 5. *Let $x = (x_1,\dots,x_\lambda) \in \mathrm{Sym}^\lambda(T)$, and let $w_1,\dots,w_\lambda \in T - x$. Then there exists $c \in \mathrm{Cor}(\mathrm{Sym}^\lambda(T))$ with $w_1,\dots,w_\lambda \notin E(c,x)$.*

Proof. We shall find $\Gamma_1,\dots,\Gamma_\lambda$ satisfying

i. For each $i = 1,\dots,\lambda$, either Γ_i is an embedding of I in T with $\Gamma_i(0) = x_i$, or, if x_i is a leaf, Γ_i may be the constant map of I to x_i.
ii. For each $i = 1,\dots,\lambda$ there is some $j \in \{1,\dots,\lambda\}$ with $w_i \in \Gamma_j(I)$.
iii. The direction is consistent. That is, if $\Gamma_i(a_1) = \Gamma_j(b_1) \neq \Gamma_i(a_2) = \Gamma_j(b_2)$ and $a_1 > a_2$ then $b_1 > b_2$.

iv. For each $i = 1, \dots, \lambda$, the end point $\Gamma_i(1)$ is a leaf.

Once we find $\Gamma_1, \dots, \Gamma_\lambda$ with these properties, we can choose

$$c = (\Gamma_1(1), \dots, \Gamma_\lambda(1))\,.$$

By iv, c is indeed a corner. For each $i = 1, \dots, \lambda$ the set $J = \{j \in \{1, \dots, \lambda\} : w_i \in \Gamma_j(I)\}$ is not empty by ii. If $j \in J$ and t is the connected component of $T - w_i$ containing x_j, then $t \cap \{c_j : j \in J\} = \emptyset$ by iii. Thus t has more elements of x than elements of c. So $w_i \notin E(c, x)$.

Of course, there exist $\Gamma_1, \dots, \Gamma_\lambda$ satisfying i and ii. For example, we can take $\Gamma_i = \Gamma_{x_i w_i}$. By permuting $w_1, \dots, w_\lambda$, we can also achieve iii. Among all choices of $\Gamma_1, \dots, \Gamma_\lambda$ satisfying i, ii and iii, we choose the ones with $\Gamma_1(I)$ maximal to inclusion. Then $\Gamma_1(1)$ must be a leaf. Among all these we choose the ones with $\Gamma_2(I)$ maximal to inclusion. We continue in this way until we achieve iv. ∎

Note that this lemma, together with Theorem 4, gives us a nice corollary that we will need later:

Corollary 3. *Let T be a tree, λ a positive integer number and let $x \in \mathrm{Sym}^\lambda(T)$. Suppose that for each corner $c \in \mathrm{Cor}(\mathrm{Sym}^\lambda(T))$ one representing connected component of $E(c, x)$ is chosen. Then we can choose $\lambda + 1$ corners with disjoint representing connected components.*

4. KKM-like theory for trees

Theorem 9. *(KKM theorem for trees) Let T be a tree and $\lambda \geq 1$ an integer number. For every corner $c \in \mathrm{Cor}(\mathrm{Sym}^\lambda(T))$ let A_c be an open set in $\mathrm{Sym}^\lambda(T)$. If each set $W \subseteq \mathrm{Cor}(\mathrm{Sym}^\lambda(T))$ satisfies $\mathrm{co}(W) \subseteq \bigcup_{c \in W} A_c$ then*

$$\bigcap_{c \in \mathrm{Cor}(\mathrm{Sym}^\lambda(T))} A_c \neq \emptyset\,.$$

Proof. Let $n = |\mathrm{Cor}(\mathrm{Sym}^\lambda(T))| = \binom{|\mathrm{L}(T)| + \lambda - 1}{\lambda}$ and let $\Delta = \Delta_{n-1}$. Recall that for every $k = 0, 1, \dots, n$ we define the k-skeleton of Δ to be the union of the convex hulls of all sets of k vertices of Δ. We will construct a continuous mapping $\rho : \Delta \to \mathrm{Sym}^\lambda(T)$. First define ρ on the 0-skeleton of Δ by assigning to each of the n vertices of Δ a different corner of $\mathrm{Sym}^\lambda(T)$. We now use the fact that $\mathrm{co}(W)$ is contractible for every subset W of $\mathrm{Cor}(\mathrm{Sym}^\lambda(T))$. We postpone the proof of that for later in this section. (See Theorem 11.) We use this for sets W of cardinality 2 to extend ρ to the 1-skeleton of Δ

with $\rho(\mathrm{co}(\{g_1,g_2\}))\subseteq \mathrm{co}(\{\rho(g_1),\rho(g_2)\})$ for every pair of vertices g_1,g_2 of Δ. We go on in this way, to extend ρ to the 2-skeleton, the 3-skeleton and eventually the entire Δ with

$$\rho(\mathrm{co}(V))\subseteq \mathrm{co}(\rho(V))$$

for every set V of vertices of Δ.

We now apply the KKM theorem on Δ with $A_g=\rho^{-1}(A_{\rho(g)})$ for every vertex g of Δ, and get

$$\bigcap_{c\in\mathrm{Cor}(\mathrm{Sym}^{\lambda}(T))} A_c \supseteq \rho\Big(\bigcap_{g\in G_n} A_g\Big)\neq\emptyset\,. \qquad \blacksquare$$

Of course, since the KKM theorem also works for closed sets, so does the KKM theorem for trees.

Theorem 10. *Let $d,\lambda_1,\dots,\lambda_d$ be a positive integer numbers and $T_1,\dots,T_d$ be disjoint trees. For every $i=1,\dots,d$ and every corner $c\in\mathrm{Cor}(\mathrm{Sym}^{\lambda_i}(T_i))$ let A_c be an open set in $\prod_i\mathrm{Sym}^{\lambda_i}(T_i)$. If each d sets $W_i\subseteq\mathrm{Cor}(\mathrm{Sym}^{\lambda_i}(T_i))$ $(i=1,\dots,d)$ satisfy*

$$\prod_i \mathrm{co}(W_i)\subseteq\bigcup_i\bigcup_{c\in W_i}A_c$$

then for some $i\in\{1,\dots,d\}$ we have

$$\bigcap_{c\in\mathrm{Cor}(\mathrm{Sym}^{\lambda_i}(T_i))}A_c\neq\emptyset\,.$$

Proof. The proof is similar to that of Theorem 9. For every $i=1,\dots,d$, let $n_i=|\mathrm{Cor}(\mathrm{Sym}^{\lambda_i}(T_i))|-1=\binom{|\mathrm{L}(T_i)|+\lambda_i-1}{\lambda_i}-1$ and let S_i be an n_i-dimensional simplex with set of vertices V_i. As in Theorem 9, we construct a continuous mapping $\rho_i:S_i\to\mathrm{Sym}^{\lambda_i}(T_i)$ with

$$\rho_i(\mathrm{co}(V))\subseteq\mathrm{co}(\rho_i(V))$$

for every set V of vertices of S_i. We then define $\rho:\prod_i S_i\to\prod_i\mathrm{Sym}^{\lambda_i}(T_i)$ by $\rho(x_1,\dots,x_d)=(\rho_1(x_1),\dots,\rho_d(x_d))$.

We now apply Theorem 2 on $\prod_i S_i$ with $A_g=\rho^{-1}(A_{\rho_i(g)})$ for every $i=1,\dots,d$ and every vertex g of S_i, and get

$$\bigcap_{c\in\mathrm{Cor}(\mathrm{Sym}^{\lambda_i}(T_i))}A_c\supseteq\rho\Big(\bigcap_{g\in V_i}A_g\Big)\neq\emptyset$$

for some $i\in\{1,\dots,d\}$. $\blacksquare$

Again, we can use closed rather than open sets. To complete the proofs of the above two theorems we need the following theorem:

Theorem 11. *Let T be a tree, $\lambda \geq 1$ an integer number and $W \subseteq \mathrm{Cor}(\mathrm{Sym}^\lambda(T))$; then $\mathrm{co}_{T\lambda}(W)$ is contractible.*

Proof. Let $P = \mathrm{co}_{T\lambda}(W), n = |\mathrm{Sin}(T)|$. The proof will be by induction on $\lambda + n$.

i. If $\lambda = 1$, then P is just the minimal connected set containing W and is therefore contractible.
ii. If $n = 0$, that is, T is homeomorphic to I, then P is the convex hull of W according to the usual definition, and again P is contractible.
iii. If there exists a leaf l that appears in all the corners in W, we consider the embedding $\phi_l : \mathrm{Sym}^{\lambda-1}(T) \to \mathrm{Sym}^\lambda(T)$, defined as above by $\phi_l((x_1,\ldots,x_{\lambda-1})) = (x_1,\ldots,x_{\lambda-1},l)$.
We first note that P is in the image of ϕ_l. If it weren't, there would exist some $x \in P$ so that l is not an element of x. We could then find some $z \in T$ which is closer to l than all the elements of x (in the sense that the connected component of $T - z$ containing l, would contain no element of x). But such z would distinguish x from every $c \in W$, with contradiction to the definition of P.
Thus P is in the image of the embedding ϕ_l and is therefore homeomorphic to $\phi_l^{-1}(P)$. We shall show that $\phi_l^{-1}(P) = \mathrm{co}_{\lambda-1}(\phi_l^{-1}(W))$, which is contractible by induction hypothesis.
Indeed, for every x, c in the image of ϕ_l, write $x' = \phi_l^{-1}(x), c' = \phi_l^{-1}(c)$, and then every $z \in T$ distinguishes x form c if and only if it distinguishes x' from c'. So $E(c,x) = E(c',x') \setminus \{l\}$ and $C(c,x) = C(c',x') \setminus \{l\}$ and we have

$$\begin{aligned}\phi_l^{-1}(P) &= \phi_l^{-1}\left(\left\{x \in \mathrm{Sym}^\lambda(T) : \bigcup_{c \in W} C(c,x) = T - x\right\}\right) \\ &= \left\{x' \in \mathrm{Sym}^{\lambda-1}(T) : \bigcup_{c' \in \phi_l^{-1}(W)} C(c',x') = T - x\right\} \\ &= \mathrm{co}_{\lambda-1}\left(\phi_l^{-1}(W)\right).\end{aligned}$$

iv. If none of the above three occurs, let r be a regular point in T and let s be a singularity point so that $co_1(\{r,s\})$ is maximal to inclusion. Let T' be the closure of the connected component of $T - s$ containing r. Then s is a leaf of T' and by the maximality of $co_1(\{r,s\})$, all other singularity points of T are in T', so $\mathrm{Sin}(T') = \mathrm{Sin}(T) \setminus \{s\}$.
Let $l_1,\ldots,l_k$ be the leaves of T which are not in T' and let $\Gamma_i = \Gamma_{l_i s}$ where $i = 1,\ldots,k$. We can now write T as the disjoint union of T' and the images $\Gamma_i([0,1))$ where $i = 1,\ldots,k$.

Let $f:T\times I\to T$ be defined by $f(\Gamma_i(a),b)=\Gamma_i(max\{a,b\})$ for $a,b\in I$ and $f(z,b)=z$ when $z\in T'$. It is easy to see that f is well defined and is a deformative retraction of T on T'. Recall the definition of $\tilde{f}:\mathrm{Sym}^{\lambda}(T)\times I\to\mathrm{Sym}^{\lambda}(T)$ by $\tilde{f}((x_1,\dots,x_\lambda),b)=(f(x_1,b),\dots,f(x_\lambda,b))$.
In order to show that P is contractible, we need to show that $\tilde{f}(P\times I)\subseteq P$, and that $\tilde{f}(P\times\{1\})$ is contractible.
We start by showing that $\tilde{f}(P\times I)\subseteq P$. Let $x\in P,b\in I,x'=\tilde{f}(x,b)$ and we want to show that $x'\in P$. In other word, we want to show that for every connected component t' of $T-x'$ we can find some $c\in W$ with $t'\subseteq C(c,x')$. Let t' be such a connected component of $T-x'$.
One possibility is that t' contains some element x_i of x. Then $f(x_i,b)$ is an element of x'. But t' doesn't contain any element of x', so $f(x_i,b)\neq x_i$ and thus $x_i=\Gamma_j(a)$ and $t'=\Gamma_j([0,b))$ for some $j\in\{1,\dots,k\}$ and $a\in[0,b)$. Let now $c\in W$ be such that l_j does not appear in c. (We dealt above with the case that such c doesn't exist.) Then $t'\subseteq C(c,x')$.
The other possibility is that t' doesn't contain any element of x, so there exists some connected component t of $T-x$ with $t'\subseteq t$. Let $c\in W$ be such that $t\subseteq C(c,x)$. (We can find such c by the way P was defined.) The function $g(a)=f(x,a)$ for $a\in[0,b]$ gives a path from x to x' in $\mathrm{Sym}^{\lambda}(T\setminus t')$, so x,x' and c are all in the same connected component of $\mathrm{Sym}^{\lambda}(T\setminus t')$. Therefore $t'\subseteq C(c,x')$.
We got this in both cases for any connected component t' of $T-x'$, and therefore $x'\in P$, and we proved $\tilde{f}(P\times I)\subseteq P$.
I claim now that

$$\tilde{f}(P\times\{1\})=P\cap\mathrm{Sym}^{\lambda}(T')=\mathrm{co}_{T'}\left(\tilde{f}(W\times\{1\})\right)\tag{1}$$

which is contractible by induction hypothesis.
The first equality is easy. We already proved $\tilde{f}(P\times\{1\})\subseteq P$, it is clear from the definition of f that $\tilde{f}(P\times\{1\})\subseteq\mathrm{Sym}^{\lambda}(T')$, and since $\tilde{f}(x,1)=x$ for every $x\in\mathrm{Sym}^{\lambda}(T')$, we have $P\cap\mathrm{Sym}^{\lambda}(T')\subseteq\tilde{f}(P\times\{1\})$.
We now need to prove the second equality in (1). Let $x\in\mathrm{Sym}^{\lambda}(T')$, let $z\in T'-x$, let $c\in W$ and let $c'=\tilde{f}(c,1)$. The function $g(a)=f(c,a)$ for $a\in[0,1]$ gives a path from c to c' in $\mathrm{Sym}^{\lambda}(T'-z)$, so z doesn't distinguish c from c'.
Suppose $x\in P$. For every connected component t' of $T'-x$, let t be the connected component of $T-x$ containing it. Since $x\in P$, we can find some $c\in W$ with $t\subseteq C(c,x)$, and by the previous paragraph, $t'\subseteq C(c',x)$, for $c'=\tilde{f}(c,1)$. This proves $P\cap\mathrm{Sym}^{\lambda}(T')\subseteq\mathrm{co}_{T'}(\tilde{f}(W\times\{1\}))$.
Conversely, suppose $x\in\mathrm{co}_{T'}(\tilde{f}(W\times\{1\}))$. For every connected component t of $T-x$, let $t'=t\cap T'$. If $t'=\emptyset$ then s must be an element of x and t

must contain some leaf l_i for some $i\in\{1,\dots,k\}$. In that cases $t\subseteq C(c,x)$, where c is any member of W not containing l_i. If $t'\neq\emptyset$ then let $c\in W$ be such that $t'\subseteq C(c',x)$, for $c'=\tilde{f}(c,1)$. Now one can see that either $t'=t$ or $s\in t'$, and in both case an easy argument gives $t\subseteq C(c,x)$.
This proves that $\mathrm{co}_{T'}(\tilde{f}(W\times\{1\}))\subseteq P$, and this ends the proof of (1). So by induction hypothesis $\tilde{f}(P\times\{1\})$ is contractible, and hence so is P. ∎

We can now prove the tree analog of Corollary 1.

Theorem 12. *Let T be a tree with some measure defined on it (so that open sets are measurable). Let λ be a positive integer number and let f : $\mathrm{Sym}^\lambda(T)\to \mathrm{L}^1(T)$ be a continuous function. Then there exists some $\bar{x}\in \mathrm{Sym}^\lambda(T)$, so that every connected component t of $T-\bar{x}$ satisfies*

$$\left|\int_t f(\bar{x})(z)dz\right|\leq\frac{1}{1+\lambda}\|f(\bar{x})\|\,.$$

Proof. For $x\in\mathrm{Sym}^\lambda(T)$ and measurable $t\subseteq T$ we write $\alpha(x)=\frac{1}{1+\lambda}\|f(x)\|$ and $\beta(t,x)=|\int_t f(x)(z)dz|$.

For each $c\in\mathrm{Cor}(\mathrm{Sym}^\lambda(T))$ let A_c be the set of all points $x\in\mathrm{Sym}^\lambda(T)$ such that some connected component t of $E(c,x)$ satisfies $\beta(t,x)>\alpha(x)$.

Let us first verify that the sets A_c are open. Let $x\in A_c$ and let t be a connected component of $E(c,x)$ satisfying $\beta(t,x)>\alpha(x)$. Let t' be a closed connected subset of t satisfying $\beta(t',x)>\alpha(x)$, and write $\varepsilon=\beta(t',x)-\alpha(x)$. We can then find a neighborhood N_x of x in $\mathrm{Sym}^\lambda(T)$, so that for all $y\in N_x$, we have $t'\subset E(c,y)$ and $\|f(x)-f(y)\|<\frac{\varepsilon}{2}$. We then have $N_x\subseteq A_c$ and thus A_c is open.

Consider now the case that Theorem 9 can be applied for the sets A_c and we have some

$$x\in\bigcap_{c\in\mathrm{Cor}(\mathrm{Sym}^\lambda(T))}A_c\,.$$

This means that for every $c\in\mathrm{Cor}(\mathrm{Sym}^\lambda(T))$ there is some connected component t_c of $E(c,x)$ satisfying $\beta(t_c,x)>\alpha(x)$. By Corollary 3 we can choose corners $c_1,\dots,c_{\lambda+1}$ so that $t_{c_1},\dots,t_{c_{\lambda+1}}$ are disjoint. We then have

$$\sum_{i=1}^{\lambda+1}\beta(t_{c_i},x)>(\lambda+1)\alpha(x)\,.$$

But this means

$$\left|\int_{\bigcup_{i=1}^{\lambda+1}t_{c_i}}f(x)(z)dz\right|>\int_T|f(x)(z)|dz\,.$$

A contradiction.

Thus the condition of Theorem 9 cannot hold, and we can find $W \subseteq \mathrm{Cor}(\mathrm{Sym}^\lambda(T))$ and $\bar{x} \in \mathrm{co}(W) \setminus \bigcup_{c\in W} A_c$.

By the definition of $\mathrm{co}(W)$, every connected component t of $T-\bar{x}$ is also a connected component of $E(c,\bar{x})$ for some $c\in W$, and since $\bar{x}\notin A_c$, we have $\beta(t,\bar{x})\leq\alpha(\bar{x})$. ∎

This theorem can be generalized even further, so that it may be applied to a Cartesian product of the symmetric powers of several trees.

Theorem 13. *Let $d,\lambda_1,\dots,\lambda_d$ be positive integer numbers and let $T_1,\dots,T_d$ be trees with some measure defined on each of them (so that open sets are measurable). For each $i=1,\dots,d$, let $f_i : \prod_i \mathrm{Sym}^{\lambda_i}(T_i) \to \mathrm{L}^1(T_i)$ be a continuous function. Then there exists some $\bar{x}=(\bar{x}_1,\dots,\bar{x}_d)\in\prod_i \mathrm{Sym}^{\lambda_i}(T_i)$, so that every $i=1,\dots,d$ and every connected component t of $T_i-\bar{x}_i$ satisfy*

$$\left|\int_t f_i(\bar{x})(z)dz\right| \leq \frac{1}{1+\lambda_i}\|f_i(\bar{x})\|\,.$$

Proof. For $x\in\prod_i \mathrm{Sym}^{\lambda_i}(T_i)$ and measurable $t\subseteq T_i$ $(i=1,\dots,d)$ we write $\alpha_i(x)=\frac{1}{1+\lambda_i}\|f_i(x)\|$ and $\beta_i(t,x)=|\int_t f_i(x)(z)dz|$.

For every $i=1,\dots,d$ and each $c\in\mathrm{Cor}(\mathrm{Sym}^{\lambda_i}(T_i))$ let A_c be the set of all points $x=(x_1,\dots,x_d)\in\prod_i \mathrm{Sym}^{\lambda_i}(T_i)$ such that some connected component t of $E(c,x_i)$ satisfies $\beta_i(t,x)>\alpha_i(x)$.

We may verify that the sets A_c are open in a way much similar to the one in the previous proof.

Consider now the case that Theorem 10 can be applied for the sets A_c and we have some

$$x=(x_1,\dots,x_d)\in\bigcap_{c\in\mathrm{Cor}(\mathrm{Sym}^{\lambda_i}(T_i))} A_c$$

for some $i\in\{1,\dots,d\}$. This means that for every $c\in\mathrm{Cor}(\mathrm{Sym}^{\lambda_i}(T_i))$ there is some connected component t_c of $E(c,x_i)$ satisfying $\beta_i(t_c,x)>\alpha_i(x)$. By Corollary 3 we can choose corners $c_1,\dots,c_{\lambda_i+1}$ so that $t_{c_1},\dots,t_{c_{\lambda_i+1}}$ are disjoint. We then have

$$\sum_{j=1}^{\lambda_i+1}\beta_i(t_{c_j},x) > (\lambda_i+1)\alpha(x)\,.$$

But this means

$$\int_{\bigcup_{j=1}^{\lambda+1} t_{c_j}} |f_i(x)(z)dz| > \int_{T_i} |f_i(x)(z)|dz\,,$$

a contradiction.

Thus the condition of Theorem 10 cannot hold, and we can find d sets $W_i \subseteq \mathrm{Cor}(\mathrm{Sym}^{\lambda_i}(T_i))$ $(i=1,\dots,d)$ and a point

$$\bar{x} = (\bar{x}_1, \dots, \bar{x}_d) \in \prod_i \mathrm{co}(W_i) \setminus \bigcup_i \bigcup_{c \in W_i} A_c \,.$$

By the definition of $\mathrm{co}(W_i)$, every connected component t of $T_i - \bar{x}_i$ is also a connected component of $E(c, \bar{x}_i)$ for some $c \in W_i$, and since $\bar{x} \notin A_c$, we have $\beta_i(t, \bar{x}) \le \alpha_i(\bar{x})$. This is true for every $i=1,\dots,d$. ∎

5. Combinatorial applications

First we shall show here a topological proof of Theorem 4. Of course, we are not allowed to use here Corollary 3 and Theorem 12, since they make use of Theorem 4. For that reason, we'll be forced to use $C(c,x)$ instead of $E(c,x)$.

Proof of Theorem 4. Let T be a tree and let $\mathcal{H}$ be a finite family of connected subsets of T. We may assume without loss of generality that those subsets are open. If $\tau(\mathcal{H})=1$ obviously $\nu(\mathcal{H})=1$. Otherwise, let $\lambda = \tau(\mathcal{H}) - 1$. For each corner $c \in \mathrm{Cor}(\mathrm{Sym}^{\lambda}(T))$, let

$$A_c = \left\{ x \in \mathrm{Sym}^{\lambda}(T) : (\exists h \in \mathcal{H})(h \subseteq C(c,x)) \right\} \,.$$

By Lemma 4 the sets A_c are closed.

For every $W \subseteq \mathrm{Cor}(\mathrm{Sym}^{\lambda}(T))$ and $x \in \mathrm{co}(W)$ we can find by the definition of $\tau(\mathcal{H})$, some $h \in \mathcal{H}$ containing no element of x. By the definition of $\mathrm{co}(W)$, we have $h \subseteq C(c,x)$ for some $c \in W$. Hence $x \in A_c$ and we have $\mathrm{co}(W) \subseteq \bigcup_{c \in W} A_c$.

We can now apply Theorem 9, so there exists $x \in \mathrm{Sym}^{\lambda}(T)$ in the intersection of all sets A_c. This doesn't necessarily mean that every connected component of $T - x$ contains some $h \in \mathcal{H}$, but by applying Lemma 5 repeatedly, we can find $\lambda + 1$ connected components of $T - x$ containing a member of $\mathcal{H}$. This implies $\nu(\mathcal{H}) \ge \lambda + 1 = \tau(\mathcal{H})$. ∎

For the proofs of Theorem 7 and Theorem 8 we need some basic hypergraph terminology and some tools developed by Kaiser in [1]. (The tools were developed for intervals, we will make some natural alterations to apply them for trees.)

If a hypergraph $\mathcal{H}$ is defined on a set V, we call the elements of V *vertices*, and the elements of $\mathcal{H}$ *edges*. (One should not confuse vertices of a hypergraph with vertices of a simplex.) The *rank* of $\mathcal{H}$ is the suprimum of the cardinalities of the edges in it. We say that $\mathcal{H}$ is *d-partite* (or *bipartite* if $d=2$) if we can write V as a disjoint union $V=V_1\cup\cdots\cup V_d$ so that $|h\cap V_i|=1$ for every $h\in\mathcal{H}$ and $i=1,\ldots,d$. An *edge weight function* on $\mathcal{H}$ is a function $w:\mathcal{H}\to[0,\infty)$ with a finite nonempty support. Every edge weight function defines a *vertex weight function* $w':V\to[0,\infty)$ by $w'(v)=\sum_{h\ni v}w(h)$.

We write

$$\nu^*(\mathcal{H}) = \sup_w \frac{\sum_{h\in\mathcal{H}} w(h)}{\max_{v\in V} w'(v)}$$

where w ranges over all edge weight functions on $\mathcal{H}$.

Theorem 14 (Kőnig, Lovász, Füredi). *Let $\mathcal{H}$ be a hypergraph with a finite rank $d\geq 2$. Then $\frac{\nu^*(\mathcal{H})}{\nu(\mathcal{H})}\leq d-1+\frac{1}{d}$. This bound can be improved to $\frac{\nu^*(\mathcal{H})}{\nu(\mathcal{H})}\leq d-1$ in the case that $\mathcal{H}$ is d-partite or in the case that $d>2$ and $\mathcal{H}$ contains no copy of the projective plane of order $d-1$.*

For $d>2$ the theorem is due to Füredi [6]. For $d=2$ in general, the theorem is due to Lovász [7]. The bipartite case is the well known theorem of Kőnig.

Let T be a tree, λ a positive integer number, $\mathcal{H}$ a hypergraph on T and $x\in\mathrm{Sym}^\lambda(T)$. We define V_x to be the set of all connected components of $T-x$. For every $h\in\mathcal{H}$ we define the *Kaiser image* of h by $K(h,x)=\{t\in V_x : h\cap t\neq\emptyset\}$. Then the *Kaiser image of* $\mathcal{H}$ is the hypergraph on V_x defined by $K(\mathcal{H},x)=\{K(h,x):h\in\mathcal{H},\ h\subseteq T-x\}$. Let us fix some metric on T agreeing with the topology on it. For every closed set h let $\mathrm{dist}(h,x)$ be the minimal distance from a point in h to a point in x. Note that for a fixed h, this is a continuous function on $\mathrm{Sym}^\lambda(T)$. If $\mathcal{H}$ is a finite hypergraph with closed edges and $K(\mathcal{H},x)\neq\emptyset$ we can define an edge weight function on $K(\mathcal{H},x)$ by

$$w_x(e) = \sum_{h:K(h,x)=e} \mathrm{dist}(h,x)\,.$$

(Here we deviate a bit from Kaiser's methods. Kaiser used suprimum instead of sum here. However, using sum seems to work better when we pass from intervals to trees.)

Note that for every $t\in V_x$ we have $w'_x(t)=\sum\mathrm{dist}(h,x)$, where the sum runs over all $h\in\mathcal{H}$ with at least one connected component contained in t.

Similarly, let $T_1,\ldots,T_d$ be disjoint trees, $\lambda_1,\ldots,\lambda_d$ positive integers, $\mathcal{H}$ a hypergraph on $F=T_1\cup\cdots\cup T_d$ and $x\in\prod_i\mathrm{Sym}^{\lambda_i}(T_i)$. We define V_x to be the set of all connected components of $F-x$. For every $h\in\mathcal{H}$ we define the Kaiser image of h by $K(h,x)=\{t\in V_x:h\cap t\neq\emptyset\}$. Then the Kaiser image of $\mathcal{H}$ is the hypergraph on V_x defined by $K(\mathcal{H},x)=\{K(h,x):h\in\mathcal{H},\ h\subseteq F-x\}$.

Again, we fix some metric on $T_1,\dots,T_d$ agreeing with the topology on them and for every closed set h we let $\mathrm{dist}(h,x)$ be the minimal distance from a point in h to a point in x. We define w_x as before.

If $\mathcal{H}$ is a d-tree hypergraph then clearly $K(\mathcal{H},x)$ is d-partite.

Also, $\nu(K(\mathcal{H},x))\le\nu(\mathcal{H})$ and $\nu^*(K(\mathcal{H},x))\le\nu^*(\mathcal{H})$ in both cases.

We will now prove the tree analogue of Lemma 2.4 in [1]:

Lemma 6. *Let $\mathcal{H}$ be a homogeneous finite d-tree hypergraph with closed edges on a tree T, and let $\lambda<\tau(\mathcal{H})$ be a positive integer number. Then there exists some $\bar{x}\in\mathrm{Sym}^\lambda(T)$ with $\nu^*(K(\mathcal{H},\bar{x}))\ge\frac{\lambda+1}{d}$.*

Proof. We fix a measure on T, where all open sets are measurable, and every connected component of an element of $\mathcal{H}$ has a positive measure. For each $h\in\mathcal{H}$, let g_h be some bounded non-negative measurable function on T, vanishing outside h and with integral 1 on every connected component of h. (For example, we can take g_h to be 0 outside of h and on every connected component of h have a constant value of $\frac{1}{m}$, where m is the measure of the connected component.)

We define a function $f:\mathrm{Sym}^\lambda(T)\to\mathrm{L}^1(T)$ by $f(x)=\sum_{h\in\mathcal{H}}\mathrm{dist}(h,x)g_h$.

Clearly, f is continuous, and by Theorem 12 we have some $\bar{x}\in\mathrm{Sym}^\lambda(T)$ with $\int_t f(\bar{x})(z)dz\le\frac{1}{1+\lambda}\|f(\bar{x})\|$ for every connected component t of $T-\bar{x}$.

Consider now the hypergraph $K(\mathcal{H},\bar{x})$ on the set $V_{\bar{x}}$ with the weight w as defined above. For every $t\in V_{\bar{x}}$ we have $\int_t f(\bar{x})(z)dz=\sum_{h\in\mathcal{H}}\mathrm{dist}(h,\bar{x})\int_t g_h(z)dz\ge w'(t)$ and also we have $\|f(\bar{x})\|=\sum_{h\in\mathcal{H}}\mathrm{dist}(h,\bar{x})\|g_h\|=d\sum_{e\in K(\mathcal{H},\bar{x})}w(e)$. Therefore

$$\nu^*(K(\mathcal{H},\bar{x}))\ge\frac{\sum_{e\in K(\mathcal{H},\bar{x})}w(e)}{\max_{t\in V_{\bar{x}}}w'(t)}\ge\frac{\lambda+1}{d}\,.\qquad\blacksquare$$

Proof of Theorem 8. Take $\lambda=\tau(\mathcal{H})-1$ and let $\bar{x}\in\mathrm{Sym}^\lambda(T)$ be such that $\nu^*(K(\mathcal{H},\bar{x}))\ge\frac{\lambda+1}{d}$. Then by Theorem 14 we have

$$\tau(\mathcal{H})=\lambda+1\le d\nu^*(K(\mathcal{H},\bar{x}))\le(d^2-d+1)\nu(K(\mathcal{H},\bar{x}))\le(d^2-d+1)\nu(\mathcal{H})$$

and the coefficient (d^2-d+1) can be replaced by (d^2-d) if $d>2$ and there is no projective plane of order $d-1$. $\blacksquare$

Lemma 7. *Let $\mathcal{H}$ be a finite d-tree hypergraph on $F=T_1\cup\dots\cup T_d$, and let $\lambda_1,\dots,\lambda_d$ be positive integer numbers with sum less than $\tau(\mathcal{H})$. Then there exists some $\bar{x}\in\prod_i\mathrm{Sym}^{\lambda_i}(T_i)$ with $\nu^*(K(\mathcal{H},\bar{x}))\ge min\lambda_i+1$.*

Proof. Again, we may assume that the sets in $\mathcal{H}$ are closed, and we fix a measure on each of $T_1,\dots,T_d$, where all open sets are measurable, and every connected component of an element of $\mathcal{H}$ has a positive measure. For each $i=1,\dots,d$ and each $h\in\mathcal{H}$, let g_{hi} be some bounded non-negative measurable function on T_i, vanishing outside $h\cap T_i$ and with norm 1 in $\mathrm{L}^1(T_i)$.

We define functions $f_i:\prod_i \mathrm{Sym}^{\lambda_i}(T_i)\to \mathrm{L}^1(T_i)$ by $f_i(x)=\sum_{h\in\mathcal{H}}\mathrm{dist}(h,x)g_{hi}$.

Clearly, f is continuous, and by Theorem 13 we have some $\bar{x}=(\bar{x}_1,\dots,\bar{x}_d)\in\prod_i \mathrm{Sym}^{\lambda_i}(T_i)$ with $\int_t f_i(\bar{x})(z)dz\le\frac{1}{1+\lambda_i}\|f_i(\bar{x})\|$ for every $i=1,\dots,d$ and every connected component t of $T_i-\bar{x}_i$.

Consider now the hypergraph $K(\mathcal{H},\bar{x})$ on the set $V_{\bar{x}}$ with the weight w as defined above. For every $t\in V_{\bar{x}}$ we have $\int_t f(\bar{x})(z)dz=\sum_{h\in\mathcal{H}}\mathrm{dist}(h,\bar{x})\int_t g_{hi}(z)dz=w'(t)$ (where $t\subseteq T_i$), and also for every $i=1,\dots,d$ we have $\|f_i(\bar{x})\|=\sum_{h\in\mathcal{H}}\mathrm{dist}(h,\bar{x})\|g_{hi}\|=\sum_{e\in K(\mathcal{H},\bar{x})}w(e)$. Therefore

$$\nu^*(K(\mathcal{H},\bar{x}))\ge\frac{\sum_{e\in K(\mathcal{H},\bar{x})}w(e)}{\max_{t\in V_{\bar{x}}}w'(t)}\ge\min\lambda+1\,. \qquad \blacksquare$$

Proof of Theorem 7. We assume for contradiction that $\tau(\mathcal{H})>(d^2-d)\nu(\mathcal{H})$. Take $\lambda_1=\cdots=\lambda_d=(d-1)\nu(\mathcal{H})$. By the assumption, $\tau(\mathcal{H})>\sum_i\lambda_i$, so we may apply Lemma 7 and find $\bar{x}\in\prod_i\mathrm{Sym}^{\lambda_i}(T_i)$ such that

$$\nu^*(K(\mathcal{H},\bar{x}))\ge\min\lambda_i+1>(d-1)\nu(\mathcal{H})\ge(d-1)\nu(K(\mathcal{H},\bar{x}))\,.$$

But $K(\mathcal{H},\bar{x})$ is d-partite, so we have a contradiction to Theorem 14. $\blacksquare$

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